

Filter Design for Positive T-S Fuzzy Continuous-Time Systems with Time Delay Using Piecewise-Linear Membership Functions

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Abstract—This work focuses on the filtering problem and stability analysis for positive Takagi-Sugeno (T-S) fuzzy systems with time delay under L_1 -induced performance. Due to the importance of estimation of system states but the few filter design results on positive nonlinear systems, it is an attractive and meaningful topic well worth studying. In order to fully exploit and take advantage of the positivity of positive T-S fuzzy systems, many commonly used methods, for instance free-weighting matrix approach and similarity transformation are probably not suitable for positive systems. To address the hard-nut-to-crack problem, an auxiliary variable is introduced so that the augmentation approach can be employed to carry out the positivity and stability analysis of filtering error systems. In addition, another obstacle that cannot be ignored is the existence of non-convex terms in the stability and positivity conditions. For getting around this barrier, some iterative linear matrix inequality (ILMI) algorithms have been proposed in the literature. However, considering the weakness that these methods cannot guarantee the convergence to a numerical solution and the iterative process is exhaustive, we present an effective matrix decoupling method to convert the non-convex conditions into convex ones in this paper. Furthermore, a linear co-positive Lyapunov function which incorporates the positivity of system states and time delay at the same time is chosen so that the positivity characteristic of filtering error systems can be captured further. However, because of plenty of valuable information of membership functions (MFs) being ignored, hence, the obtained results are conservative. For the sake of relaxing the conservativeness, the advanced piecewise-linear membership functions (PLMFs) approximate method is utilized to facilitate the stability and positivity analysis. Therefore, the relaxed stability and positivity conditions which are cast as sum of squares (SOS) are obtained and can be solved numerically. Finally, the effectiveness of the designed fuzzy filtering strategy with satisfying L_1 -induced performance are demonstrated by a simulation example.

Index Terms—Positive T-S fuzzy systems, filter design, L_1 -induced performance, sum of squares (SOS).

I. INTRODUCTION

IN the last decade or so, researchers paid more attention to positive systems which have some special properties. For example, the state variables and system outputs will stay in

the positive quadrant if the initial conditions are non-negative [1]–[3]. Such systems are very close to our daily life, such as, in biological and physiological field: metabolic systems; in ecology and population dynamics: the human, animal and plant populations; in communications: data packets flowing in a network; and in compartmental systems: pharmacokinetics, epidemiology and so on. Therefore, it is very important and valuable to have a deep understanding for positive systems.

Different from general systems, positive systems are defined on cones rather than linear spaces. This kind of unique and elegant positivity will make numerous well-established results for general systems not applicable to positive systems. Therefore, an increasing number of researchers have focused on positive systems from different perspectives, such as the reachability and controllability [4]–[6], the realization [7], [8], the control synthesis, stability analysis and optimal control for positive systems [9]–[12]. Furthermore, different control strategies have been employed for positive systems, for instance, state feedback control strategy [13], [14], static output feedback control strategy [15], [16], observer design [17], positive filter design [18], [19] and so on.

The filter design for general systems has been attracting researchers for decades and rich results have been achieved [20]–[23]. In view of the various filtering results in the literature, we notice that the L_1 filtering and H_∞ filtering [24], [25] are more popular than Kalman filtering [26], [27] because the formers can achieve a great filtering performance without the exact knowledge of the statistics of the external noise signals [28]. Another advantage of L_1 filtering and H_∞ filtering is insensitivity to the uncertainties either in the exogenous signal statistics or in dynamic models. All of these benefits of the L_1 filtering and H_∞ filtering make them more applicable to some practical systems. However, when designing filters for positive systems, people are more concerned about the total number of the components or the maximal quantities which can be captured by L_1 filtering rather than H_∞ filtering. Thereby, from this point, the advantages of L_1 filtering are more obvious than H_∞ filtering. To the best of our knowledge, although the L_1 filtering is helpful to describe the features of positive systems, the research results are not fruitful enough at present. Hence, it is one of the motivations for us to carry out this work.

On the other hand, although there are some results relating to filter design for positive systems, it is noted in the literature that most of the works are for positive linear systems [18], [19], [29]. In [18], positive filter for continuous-time positive systems under L_1 performance was investigated, meanwhile,

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a new feature of filter was discovered firstly to guarantee the stability and L_1 -induced performance of the filter error system at the same time. The work in [19] mainly designed the event-triggered network-based ℓ_1 -gain filter for positive linear systems, in which the sufficient conditions were derived so that the positive linear filter existed with satisfying ℓ_1 -gain performance. In order to estimate the output of positive switched systems, the authors in [29] designed a positive filter based on both the upper-bound and lower-bound information of system states. Regrettably, as a matter of fact that many positive systems are of nonlinear in practical applications, such as the metabolic pathways [30], buck converter [31] and the gas-lifted oil well system [32], which means that the existing results for positive linear systems may be unsuitable for positive nonlinear systems [19], [33]. Therefore, the investigation of stability analysis and control design for positive nonlinear systems is well motivated and the achievements will expand the knowledge in the field.

Up to now, because of the complexity, high non-linearity and unique positivity constrain of positive nonlinear systems, the relevant results are less fruitful. Considering the case that only parts of the system states are available, we have investigated the static output feedback control synthesis for positive nonlinear based on fuzzy model in [34]–[36], but when all of the system states are not available, these results using output feedback control strategy cannot be applied. Meanwhile, in consideration of the influence of external disturbance for system stability, designing positive fuzzy filters for positive nonlinear systems is of great significant and meaningful.

Fuzzy control belongs to one of the intelligent control methods and has been widely applied in the past several decades [37]–[40]. Especially, the popular Takagi-Sugeno (T-S) fuzzy model [41]–[45] has obvious superiorities in approximating complicated nonlinear systems. Nevertheless, the results related to the positive T-S filter design for positive nonlinear continuous-time systems using T-S fuzzy model are relatively few. From the literature, we find the work in [46] is mainly related to positive filter design for positive T-S fuzzy systems under L_1 performance, but it aims at the discrete-time systems without time delay. While the work in this paper mainly aims at positive T-S continuous-time systems with time delay. And from the analysis technique perspective, an auxiliary variable which is helpful to construct an augmented system will be introduced to facilitate the positivity and stability study, which is totally different from the analysis in [46]. Besides, it should be pointed out that time delay which not only can result in degradation of system performance but also is able to cause instability is frequently encountered in practical systems [47], such as, chemical reaction process, network transmission and reproduction of plants and animals. Hence, it is vital to take the time delay into account when the positive filter is designed for positive T-S continuous-time systems, which makes the results have a great of theoretical value and practical significance.

As mentioned before, we know that positive systems have the elegant property that all of the state variables are non-negative values if the initial conditions are non-negative. Based on extensive literature review, it is shown that the linear co-positive Lyapunov function not only can better capture the

unique positivity of positive systems but also can reduce the difficulty of the analysis and computational burden [48]–[50]. In order to take advantage of this property, a linear co-positive Lyapunov function rather than a quadratic Lyapunov function is chosen to analyze the stability and positivity of the positive T-S fuzzy filter error systems in this paper. Up to now, as far as we know, there has been no L_1 -induced positive filter design and stability analysis reported for positive T-S continuous-time systems with time delay by using linear co-positive Lyapunov function, which motivates us to study the topic to fill this gap.

At present, some advanced membership-function-dependent (MFD) techniques, for instance, staircase membership functions (MFs) [51], polynomial MFs [34], [52] and piecewise-linear MFs (PLMFs) [53], have been proposed to facilitate relaxed stability analysis. These techniques provide great help to extract the information of MFs, such as the shape information and boundary information. However, they usually are not taken into the stability analysis of positive T-S fuzzy systems because the MFs will lead to the complexity of stability analysis and MFD analysis is still in its early research stage [54]. In addition, when this advanced technique is employed for general fuzzy systems, the MFs $w_i(\mathbf{x})$ and $m_j(\mathbf{x})$ are only related to \mathbf{x} . However, in this paper, the MFs $w_i(\mathbf{x})$ and $m_j(\mathbf{x}_f)$ are related to \mathbf{x} and \mathbf{x}_f , respectively, which means the complexity is greatly increased. Therefore, it is a very challenging task for us. On the other hand, trying to find a suitable approach to introduce the information of MFs into the stability analysis for positive T-S fuzzy filter systems incredibly stimulates us to perform the work.

In order to accomplish our purpose, there are some obstacles that require to be overcome: firstly, in order to capture the positivity of positive systems, it is more popular to analyze the stability by using the linear co-positive Lyapunov function compared with the quadratic Lyapunov function, but when using this advanced method, many useful techniques which are appropriate for quadratic Lyapunov function, such as the free-matrix approach and the similarity transformation might not appropriate for linear co-positive Lyapunov function. Therefore, in order to facilitate the positivity and stability analysis without the help of these methods, we will introduce an auxiliary variable which is in favor of the use of the augmentation approach [55]. Secondly, there are some non-convex terms in the stability conditions, which make it hard to obtain the numerical solution based on the current state of the art. To break through this barrier, some iterative linear matrix inequality (ILMI) algorithms have been proposed. But it is well known that these methods cannot guarantee the convergence to a numerical solution and the iterative process is exhaustive. Thereby, we will develop a matrix decoupling method [18] to convert the non-convex conditions into convex ones so that this problem can be dealt with skillfully. Thirdly, for better capturing the positivity of the positive systems, L_1 -induced performance index instead of H_∞ performance is considered and a linear co-positive Lyapunov function candidate instead of a quadratic Lyapunov function candidate is chosen in this paper. Finally, some important information embedded in MFs is extracted and introduced into the stability analysis by employing the advanced PMLFs approximate technique so

that the obtained stability and positivity conditions are relaxed.

The following is the arrangement of this paper. In Section II, we give a number of useful preliminaries and the positive filter design procedure. In Section III, the positivity and stability analysis is carried out for positive T-S fuzzy filter error systems. In Section IV, we show an example to verify the reliability and validity of the theoretical results. The Section V mainly shows the conclusion.

II. PRELIMINARIES

In this section, we will show some standard notations and give the positive T-S fuzzy model with time delay and the positive T-S fuzzy filter mathematically.

A. Notation

Over the course of the entire paper, the following notations are employed. For a matrix $\mathbf{M} \in \mathbb{R}^{l \times n}$ where m_{rs} denotes the element on r -th row and s -th column. $\mathbf{M} \succeq 0$, $\mathbf{M} \succ 0$, $\mathbf{M} \preceq 0$ and $\mathbf{M} \prec 0$ represent that each element m_{rs} is non-negative, positive, non-positive and negative, respectively. $\mathbf{Q}(\mathbf{x}) = \text{diag}(x_1, \dots, x_n)$ denotes that the matrix $\mathbf{Q}(\mathbf{x})$ is a diagonal matrix whose diagonal elements are x_1, \dots, x_n .

B. Positive T-S Fuzzy Model with Time Delay

The dynamics of the positive T-S fuzzy system with time delay is given as follows:

$$\begin{aligned} \text{Rule } i : & \text{ IF } \theta_1(t) \text{ is } M_1^i \text{ AND } \dots \text{ AND } \theta_\Psi(t) \text{ is } M_\Psi^i \\ \text{THEN} & \\ \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{A}_{\omega i} \tilde{\mathbf{w}}(t) + \mathbf{A}_{di} \mathbf{x}(t - \tau), \\ \mathbf{y}(t) = \mathbf{C}_i \mathbf{x}(t) + \mathbf{C}_{\omega i} \tilde{\mathbf{w}}(t) + \mathbf{C}_{di} \mathbf{x}(t - \tau), \\ \mathbf{z}(t) = \mathbf{E}_i \mathbf{x}(t) + \mathbf{E}_{\omega i} \tilde{\mathbf{w}}(t) + \mathbf{E}_{di} \mathbf{x}(t - \tau), \\ \mathbf{x}(\varrho) = \chi(\varrho), \varrho \in [-\tau, 0], \end{cases} & (1) \end{aligned}$$

where $\theta_l(t)$ is the premise variable, Ψ is a positive integer, $l \in \{1, 2, \dots, \Psi\}$; M_l^i is the fuzzy set of the i -th rule corresponding to the function $\theta_l(t)$; $\mathbf{x}(t) \in \mathbb{R}^n$, $\tilde{\mathbf{w}}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^l$ and $\mathbf{z}(t) \in \mathbb{R}^q$ are the system state vector, the disturbance signal, measurement output and the signal to be estimated, respectively; $\tau > 0$ is a given constant time-delay and $\chi(\varrho)$ is the initial function; \mathbf{A}_i , $\mathbf{A}_{\omega i}$, \mathbf{A}_{di} , \mathbf{C}_i , $\mathbf{C}_{\omega i}$, \mathbf{C}_{di} , \mathbf{E}_i , $\mathbf{E}_{\omega i}$, \mathbf{E}_{di} are the system matrices with compatible dimensions.

We get the dynamics of the positive T-S fuzzy system with time delay as follows:

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^p \mu_i(\theta(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{A}_{\omega i} \tilde{\mathbf{w}}(t) + \mathbf{A}_{di} \mathbf{x}(t - \tau)), \\ \mathbf{y}(t) = \sum_{i=1}^p \mu_i(\theta(t)) (\mathbf{C}_i \mathbf{x}(t) + \mathbf{C}_{\omega i} \tilde{\mathbf{w}}(t) + \mathbf{C}_{di} \mathbf{x}(t - \tau)), \\ \mathbf{z}(t) = \sum_{i=1}^p \mu_i(\theta(t)) (\mathbf{E}_i \mathbf{x}(t) + \mathbf{E}_{\omega i} \tilde{\mathbf{w}}(t) + \mathbf{E}_{di} \mathbf{x}(t - \tau)), \\ \mathbf{x}(\varrho) = \chi(\varrho), \varrho \in [-\tau, 0], \end{cases} \quad (2)$$

where $\mu_i(\theta(t)) = \frac{\prod_{l=1}^{\Psi} \phi_{M_l^i}(\theta(t))}{\sum_{k=1}^p \prod_{l=1}^{\Psi} \phi_{M_l^k}(\theta(t))}$, $\sum_{i=1}^p \mu_i(\theta(t)) = 1$, $\mu_i(\theta(t)) \geq 0, \forall i$ and $\mu_i(\theta(t))$ is the normalized grade of mem-

bership; $\phi_{M_l^i}(\theta(t))$ is the grade of membership corresponding to the fuzzy term M_l^i ; p is the number of the fuzzy rules.

The following assumptions and definitions are presented to support the analysis.

Assumption 1: All of the system states of a positive nonlinear system are not able to be measured.

Assumption 2: A positive nonlinear system is affected by disturbance signals and time delays.

Assumption 3: A positive nonlinear system is able to be expressed by a positive T-S fuzzy model (2).

Definition 1: [18] A system is deemed to be positive if the initial condition $\mathbf{x}(0) = \mathbf{x}_0 \succeq 0$ holds and the corresponding trajectory $\mathbf{x}(t) \succeq 0$, $\mathbf{z}(t) \succeq 0$, $\mathbf{y}(t) \succeq 0$ and $\tilde{\mathbf{w}}(t) \succeq 0$ for all $t \geq 0$ is satisfied.

Definition 2: [1] A matrix \mathbf{M} is called a Metzler matrix if its off-diagonal elements are non-negative: $m_{rs} \geq 0$, $r \neq s$.

Lemma 1: [56], [57] System (2) is a positive system if \mathbf{A}_i is a Metzler matrix, and the rest of the system matrices satisfy that all of the elements in each matrix are non-negative.

C. Positive T-S Fuzzy Filter Design

The positive T-S fuzzy filter is described by c rules of the following format:

$$\begin{aligned} \text{Rule } j : & \text{ IF } \vartheta_1(t) \text{ is } N_1^j \text{ AND } \dots \text{ AND } \vartheta_\Omega(t) \text{ is } N_\Omega^j \\ \text{THEN} & \\ \begin{cases} \dot{\mathbf{x}}_f(t) = \mathbf{A}_{fj} \mathbf{x}_f(t) + \mathbf{B}_{fj} \mathbf{y}(t), \\ \mathbf{z}_f(t) = \mathbf{C}_{fj} \mathbf{x}_f(t) + \mathbf{D}_{fj} \mathbf{y}(t), \end{cases} & (3) \end{aligned}$$

where $\mathbf{x}_f(t) \in \mathbb{R}^n$ is the filter state, $\mathbf{z}_f(t) \in \mathbb{R}^q$ is the estimated output, \mathbf{A}_{fj} , \mathbf{B}_{fj} , \mathbf{C}_{fj} , \mathbf{D}_{fj} are the positive T-S fuzzy filter gain matrices to be determined.

Then the overall positive T-S fuzzy filter is established as follows:

$$\begin{cases} \dot{\mathbf{x}}_f(t) = \sum_{j=1}^c \eta_j(\vartheta(t)) (\mathbf{A}_{fj} \mathbf{x}_f(t) + \mathbf{B}_{fj} \mathbf{y}(t)), \\ \mathbf{z}_f(t) = \sum_{j=1}^c \eta_j(\vartheta(t)) (\mathbf{C}_{fj} \mathbf{x}_f(t) + \mathbf{D}_{fj} \mathbf{y}(t)), \end{cases} \quad (4)$$

where $\eta_j(\vartheta(t)) = \frac{\prod_{\beta=1}^{\Omega} \phi_{N_\beta^j}(\vartheta(t))}{\sum_{k=1}^c \prod_{\beta=1}^{\Omega} \phi_{N_\beta^k}(\vartheta(t))}$, $\eta_j(\vartheta(t)) \geq 0$, $\sum_{j=1}^c \eta_j(\vartheta(t)) = 1$, $\forall j$ and $\eta_j(\vartheta(t))$ is the normalized grade of membership; $\phi_{N_\beta^j}(\vartheta(t))$ is the grade of membership corresponding to the fuzzy term of N_β^j .

For the sake of simplicity, we will omit t in the following analysis. That means $\mathbf{x}(t)$, $\mathbf{x}(t - \tau)$, $\mathbf{x}_f(t)$, $\mathbf{z}(t)$, $\mathbf{z}_f(t)$, $\mathbf{y}(t)$ and $\tilde{\mathbf{w}}(t)$ will be replaced by \mathbf{x} , \mathbf{x}_τ , \mathbf{x}_f , \mathbf{z} , \mathbf{z}_f , \mathbf{y} and $\tilde{\mathbf{w}}$, respectively. Meanwhile, $\mu_i(\theta(t))$ and $\eta_j(\vartheta(t))$ will be replaced by μ_i and η_j , respectively.

Defining $\zeta = [\mathbf{x}; \mathbf{x}_f - \mathbf{x}]$ and $\tilde{\mathbf{z}} = \mathbf{z}_f - \mathbf{z}$, the positive T-S fuzzy filter error system is written as follows:

$$\begin{cases} \dot{\zeta} = \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j (\hat{\mathbf{A}}_{ij} \zeta + \hat{\mathbf{A}}_{\omega ij} \tilde{\mathbf{w}} + \hat{\mathbf{A}}_{dij} \mathbf{x}_\tau), \\ \tilde{\mathbf{z}} = \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j (\hat{\mathbf{L}}_{ij} \zeta + \hat{\mathbf{L}}_{\omega ij} \tilde{\mathbf{w}} + \hat{\mathbf{L}}_{dij} \mathbf{x}_\tau), \end{cases} \quad (5)$$

where $\hat{\mathbf{A}}_{ij} = \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \mathbf{A}_{fj} + \mathbf{B}_{fj}\mathbf{C}_i - \mathbf{A}_i & \mathbf{A}_{fj} \end{bmatrix}$, $\hat{\mathbf{A}}_{\omega ij} = \begin{bmatrix} \mathbf{A}_{\omega i} \\ \mathbf{B}_{fj}\mathbf{C}_{\omega i} - \mathbf{A}_{\omega i} \end{bmatrix}$, $\hat{\mathbf{A}}_{dij} = \begin{bmatrix} \mathbf{A}_{di} \\ \mathbf{B}_{fj}\mathbf{C}_{di} - \mathbf{A}_{di} \end{bmatrix}$, $\hat{\mathbf{L}}_{ij} = \begin{bmatrix} \mathbf{D}_{fj}\mathbf{C}_i - \mathbf{E}_i + \mathbf{C}_{fj} & \mathbf{C}_{fj} \end{bmatrix}$, $\hat{\mathbf{L}}_{\omega ij} = \begin{bmatrix} \mathbf{D}_{fj}\mathbf{C}_{\omega i} - \mathbf{E}_{\omega i} \\ \mathbf{D}_{fj}\mathbf{C}_{di} - \mathbf{E}_{di} \end{bmatrix}$.

Remark 1: On the strength of Lemma 1, the positive T-S fuzzy filter (4) is positive, if \mathbf{A}_{fj} is a Metzler matrix, $\mathbf{B}_{fj} \succeq 0$, $\mathbf{C}_{fj} \succeq 0$, $\mathbf{D}_{fj} \succeq 0$ for all j .

Remark 2: Based on Lemma 1, the positive T-S fuzzy filter error system (5) is positive, if $\hat{\mathbf{A}}_{ij}$ is a Metzler matrix, $\hat{\mathbf{A}}_{\omega ij} \succeq 0$, $\hat{\mathbf{A}}_{dij} \succeq 0$, $\hat{\mathbf{L}}_{ij} \succeq 0$, $\hat{\mathbf{L}}_{\omega ij} \succeq 0$, $\hat{\mathbf{L}}_{dij} \succeq 0$, for all i, j .

For further investigating the filter design, the definition of L_1 -induced performance is given firstly in the following.

Definition 3: [58] It is seen that the system (5) has L_1 -induced performance at the level γ , if under zero initial conditions, the following inequality is satisfied

$$\|\tilde{\mathbf{z}}\|_{L_1} < \gamma \|\tilde{\mathbf{w}}\|_{L_1}, \quad (6)$$

where γ is the optimal level to be determined.

In this paper, for a given stable positive T-S fuzzy system with time delay, we aim to design a positive T-S fuzzy filter so that the stability and positivity of the positive T-S fuzzy filter error system can be ensured under L_1 -induced performance. Therefore, let us focus on this target in the following work.

III. STABILITY ANALYSIS

In this section, to facilitate the stability and positivity analysis, a novel augmented positive T-S fuzzy filter error system is given by introducing an auxiliary variable. **Furthermore, to better extract the positivity of the positive system, a linear co-positive Lyapunov function is used to analyze the stability and positivity of the positive T-S fuzzy filter error system.** In addition, the advanced PLMFs technique is considered to help reduce the conservativeness of the results.

A. The Augmented Positive T-S Fuzzy Filter Error System

In the matrices $\hat{\mathbf{A}}_{ij}$, $\hat{\mathbf{A}}_{\omega ij}$, $\hat{\mathbf{A}}_{dij}$, $\hat{\mathbf{L}}_{ij}$, $\hat{\mathbf{L}}_{\omega ij}$ and $\hat{\mathbf{L}}_{dij}$, we can find the determined matrices \mathbf{B}_{fj} , \mathbf{D}_{fj} are coupled with \mathbf{C}_i , $\mathbf{C}_{\omega i}$, and \mathbf{C}_{di} , which will make the non-convex terms in stability conditions is hard to deal with. In order to solve this problem, we try to pick out these determined matrices from $\hat{\mathbf{A}}_{ij}$, $\hat{\mathbf{A}}_{\omega ij}$, $\hat{\mathbf{A}}_{dij}$, $\hat{\mathbf{L}}_{ij}$, $\hat{\mathbf{L}}_{\omega ij}$ and $\hat{\mathbf{L}}_{dij}$. Hence, **an auxiliary variable which is shown as (8) is adopted to facilitate the analysis.** Then the system matrices of the positive T-S fuzzy filter error system are dealt with as follows:

$$\begin{aligned} \hat{\mathbf{A}}_{ij} &= \bar{\mathbf{A}}_{ij} + \mathbf{M}_j \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}, \hat{\mathbf{A}}_{\omega ij} = \bar{\mathbf{A}}_{\omega i} + \mathbf{M}_j \tilde{\mathbf{H}}_i \tilde{\mathbf{D}}, \\ \hat{\mathbf{A}}_{dij} &= \bar{\mathbf{A}}_{di} + \mathbf{M}_j \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}_d, \hat{\mathbf{L}}_{ij} = \bar{\mathbf{E}}_{ij} + \mathbf{N}_j \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}, \\ \hat{\mathbf{L}}_{\omega ij} &= \bar{\mathbf{E}}_{\omega i} + \mathbf{N}_j \tilde{\mathbf{H}}_i \tilde{\mathbf{D}}, \hat{\mathbf{L}}_{dij} = \bar{\mathbf{E}}_{di} + \mathbf{N}_j \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}_d. \end{aligned} \quad (7)$$

where $\bar{\mathbf{A}}_{ij} = \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \mathbf{A}_{fj} - \mathbf{A}_i & \mathbf{A}_{fj} \end{bmatrix}$, $\bar{\mathbf{A}}_{\omega i} = \begin{bmatrix} \mathbf{A}_{\omega i} \\ -\mathbf{A}_{\omega i} \end{bmatrix}$, $\bar{\mathbf{A}}_{di} = \begin{bmatrix} \mathbf{A}_{di} \\ -\mathbf{A}_{di} \end{bmatrix}$, $\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$, $\tilde{\mathbf{C}}_d = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix}$, $\tilde{\mathbf{H}}_i =$

$$\begin{bmatrix} \mathbf{C}_i & \mathbf{C}_{\omega i} & \mathbf{C}_{di} \end{bmatrix}, \mathbf{M}_j = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{fj} \end{bmatrix}, \mathbf{N}_j = \begin{bmatrix} \mathbf{D}_{fj} \end{bmatrix}, \bar{\mathbf{E}}_{ij} = \begin{bmatrix} \mathbf{C}_{fj} - \mathbf{E}_i & \mathbf{C}_{fj} \end{bmatrix}, \bar{\mathbf{E}}_{\omega i} = \begin{bmatrix} -\mathbf{E}_{\omega i} \end{bmatrix}, \bar{\mathbf{E}}_{di} = \begin{bmatrix} -\mathbf{E}_{di} \end{bmatrix}.$$

The auxiliary variable is chosen as follows:

$$\tilde{\psi} = \sum_{i=1}^p \mu_i (\tilde{\mathbf{H}}_i \tilde{\mathbf{C}} \zeta + \tilde{\mathbf{H}}_i \tilde{\mathbf{D}} \tilde{\mathbf{w}} + \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}_d \mathbf{x}_\tau). \quad (8)$$

Due to $\tilde{\mathbf{H}}_i \succeq 0$, $\tilde{\mathbf{C}} \succeq 0$, $\tilde{\mathbf{D}} \succeq 0$ and $\tilde{\mathbf{C}}_d \succeq 0$, we can see that each element in $\tilde{\psi}$ is non-negative. Then following on (5) and (8), **an augmented system is presented** with the definition of $\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\xi = [\zeta; \tilde{\psi}]$:

$$\begin{aligned} \mathbf{E} \dot{\xi} &= \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j (\tilde{\mathbf{A}}_{ij} \xi + \tilde{\mathbf{A}}_{\omega i} \tilde{\mathbf{w}} + \tilde{\mathbf{A}}_{di} \mathbf{x}_\tau), \\ \tilde{\mathbf{z}} &= \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j (\tilde{\mathbf{L}}_{ij} \xi + \tilde{\mathbf{L}}_{\omega i} \tilde{\mathbf{w}} + \tilde{\mathbf{L}}_{di} \mathbf{x}_\tau), \end{aligned} \quad (9)$$

where $\tilde{\mathbf{A}}_{ij} = \begin{bmatrix} \tilde{\mathbf{A}}_{ij} & \mathbf{M}_j \\ \tilde{\mathbf{H}}_i \tilde{\mathbf{C}} & -\mathbf{I} \end{bmatrix}$, $\tilde{\mathbf{A}}_{\omega i} = \begin{bmatrix} \tilde{\mathbf{A}}_{\omega i} \\ \tilde{\mathbf{H}}_i \tilde{\mathbf{D}} \end{bmatrix}$, $\tilde{\mathbf{A}}_{di} = \begin{bmatrix} \tilde{\mathbf{A}}_{di} \\ \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}_d \end{bmatrix}$, $\tilde{\mathbf{L}}_{ij} = \begin{bmatrix} \bar{\mathbf{E}}_{ij} & \mathbf{N}_j \end{bmatrix}$, $\tilde{\mathbf{L}}_{\omega i} = \begin{bmatrix} \bar{\mathbf{E}}_{\omega i} \end{bmatrix}$, $\tilde{\mathbf{L}}_{di} = \begin{bmatrix} \bar{\mathbf{E}}_{di} \end{bmatrix}$.

B. Positivity and Stability Analysis of Positive T-S Fuzzy Filter Error System

In the following, a proper Lyapunov function is chosen so that not only can the positivity of the positive system be captured, but also the time delay can be taken into account:

$$V(t) = \mathbf{p}^T \mathbf{E} \xi + \int_{t-\tau}^t \lambda^T \mathbf{F}_d \mathbf{x}(s) ds, \quad (10)$$

where $0 \preceq \lambda \in \mathbb{R}^n$ is a **vector** to be determined and $0 \preceq \mathbf{F}_d \in \mathbb{R}^{n \times n}$ is a given matrix.

The derivative of the above Lyapunov function is obtained:

$$\begin{aligned} \dot{V}(t) &= \mathbf{p}^T \mathbf{E} \dot{\xi} + \lambda^T \mathbf{F}_d \mathbf{x} - \lambda^T \mathbf{F}_d \mathbf{x}_\tau \\ &= \mathbf{p}^T \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j (\tilde{\mathbf{A}}_{ij} \xi + \tilde{\mathbf{A}}_{\omega i} \tilde{\mathbf{w}} + \tilde{\mathbf{A}}_{di} \mathbf{x}_\tau) \\ &\quad + \lambda^T \mathbf{F}_d \mathbf{x} - \lambda^T \mathbf{F}_d \mathbf{x}_\tau. \end{aligned} \quad (11)$$

Next, taking the L_1 performance index (6) into account:

$$\begin{aligned} J &= \int_0^T \|\tilde{\mathbf{z}}\|_{L_1} - \gamma \|\tilde{\mathbf{w}}\|_{L_1} dt = \int_0^T \sum_{k=1}^q \tilde{z}_k - \gamma \sum_{k=1}^m \tilde{w}_k dt \\ &= \int_0^T \sum_{k=1}^q \tilde{z}_k - \gamma \sum_{k=1}^m \tilde{w}_k + \dot{V}(t) dt + V(0) - V(T). \end{aligned} \quad (12)$$

Due to the zero initial condition, and $V(T) \rightarrow 0$ when $T \rightarrow \infty$, (12) can be treated as:

$$\begin{aligned} J &= \int_0^\infty \sum_{k=1}^q \tilde{z}_k - \gamma \sum_{k=1}^m \tilde{w}_k + \dot{V}(t) dt \\ &= \int_0^\infty \mathbf{I}_1^T \tilde{\mathbf{z}} - \gamma \mathbf{I}_2^T \tilde{\mathbf{w}} + \dot{V}(t) dt, \end{aligned} \quad (13)$$

where $\mathbf{I}_1 \in \mathbb{R}^q$ and $\mathbf{I}_2 \in \mathbb{R}^m$ are **vectors** with all of the elements being 1.

Taking (9) and (11) into (13), we have:

$$\begin{aligned}
J &= \int_0^\infty \mathbf{I}_1^T \left(\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j (\tilde{\mathbf{L}}_{ij} \xi + \tilde{\mathbf{L}}_{\omega i} \tilde{\mathbf{w}} + \tilde{\mathbf{L}}_{di} \mathbf{x}_\tau) \right) \\
&\quad + \mathbf{p}^T \left(\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j (\tilde{\mathbf{A}}_{ij} \xi + \tilde{\mathbf{A}}_{\omega i} \tilde{\mathbf{w}} + \tilde{\mathbf{A}}_{di} \mathbf{x}_\tau) \right) \\
&\quad + \lambda^T \mathbf{F}_d \mathbf{x} - \lambda^T \mathbf{F}_d \mathbf{x}_\tau - \gamma \mathbf{I}_2^T \tilde{\mathbf{w}}_d \\
&= \int_0^\infty \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left((\mathbf{I}_1^T \tilde{\mathbf{L}}_{\omega i} + \mathbf{p}^T \tilde{\mathbf{A}}_{\omega i} - \gamma \mathbf{I}_2^T) \tilde{\mathbf{w}} \right. \\
&\quad + (\mathbf{I}_1^T \tilde{\mathbf{L}}_{ij} + \mathbf{p}^T \tilde{\mathbf{A}}_{ij} + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times (n+l)}]) \xi \\
&\quad \left. + (\mathbf{I}_1^T \tilde{\mathbf{L}}_{di} + \mathbf{p}^T \tilde{\mathbf{A}}_{di} - \lambda^T \mathbf{F}_d) \mathbf{x}_\tau \right) dt, \quad (14)
\end{aligned}$$

where $\mathbf{I}_{n \times n}$ is a unit matrix with the dimensions of $n \times n$, $\mathbf{0}_{n \times (n+l)}$ is a null matrix with the dimensions of $n \times (n+l)$.

Based on (14), we can see that $J < 0$ can be ensured by the following conditions:

$$\mathbf{I}_1^T \tilde{\mathbf{L}}_{ij} + \mathbf{p}^T \tilde{\mathbf{A}}_{ij} + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times (n+l)}] \prec 0, \forall i, j; \quad (15)$$

$$\mathbf{I}_1^T \tilde{\mathbf{L}}_{\omega i} + \mathbf{p}^T \tilde{\mathbf{A}}_{\omega i} - \gamma \mathbf{I}_2^T \prec 0, \forall i; \quad (16)$$

$$\mathbf{I}_1^T \tilde{\mathbf{L}}_{di} + \mathbf{p}^T \tilde{\mathbf{A}}_{di} - \lambda^T \mathbf{F}_d \prec 0, \forall i. \quad (17)$$

For further handling the above inequalities, we have:

$$\begin{aligned}
&\mathbf{I}_1^T \tilde{\mathbf{L}}_{ij} + \mathbf{p}^T \tilde{\mathbf{A}}_{ij} + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times (n+l)}] \\
&= \mathbf{I}_1^T [\tilde{\mathbf{E}}_{ij} \quad \mathbf{N}_j] + [\mathbf{p}_1^T \quad \mathbf{p}_2^T] \begin{bmatrix} \tilde{\mathbf{A}}_{ij} & \mathbf{M}_j \\ \tilde{\mathbf{H}}_i \tilde{\mathbf{C}} & -\mathbf{I} \end{bmatrix} \\
&\quad + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times (n+l)}] \\
&= [\mathbf{I}_1^T \tilde{\mathbf{E}}_{ij} + \mathbf{p}_1^T \tilde{\mathbf{A}}_{ij} + \mathbf{p}_2^T \tilde{\mathbf{H}}_i \tilde{\mathbf{C}} + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times n}]] \\
&\quad \mathbf{I}_1^T \mathbf{N}_j + \mathbf{p}_1^T \mathbf{M}_j - \mathbf{p}_2^T, \quad (18)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_1^T \tilde{\mathbf{L}}_{\omega i} + \mathbf{p}^T \tilde{\mathbf{A}}_{\omega i} - \gamma \mathbf{I}_2^T \\
&= \mathbf{I}_1^T \tilde{\mathbf{E}}_{\omega i} - \gamma \mathbf{I}_2^T + [\mathbf{p}_1^T \quad \mathbf{p}_2^T] \begin{bmatrix} \tilde{\mathbf{A}}_{\omega i} \\ \tilde{\mathbf{H}}_i \tilde{\mathbf{D}} \end{bmatrix} \\
&= \mathbf{I}_1^T \tilde{\mathbf{E}}_{\omega i} - \gamma \mathbf{I}_2^T + \mathbf{p}_1^T \tilde{\mathbf{A}}_{\omega i} + \mathbf{p}_2^T \tilde{\mathbf{H}}_i \tilde{\mathbf{D}}, \quad (19) \\
&\mathbf{I}_1^T \tilde{\mathbf{L}}_{di} + \mathbf{p}^T \tilde{\mathbf{A}}_{di} - \lambda^T \mathbf{F}_d \\
&= \mathbf{I}_1^T \tilde{\mathbf{E}}_{di} + [\mathbf{p}_1^T \quad \mathbf{p}_2^T] \begin{bmatrix} \tilde{\mathbf{A}}_{di} \\ \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}_d \end{bmatrix} - \lambda^T \mathbf{F}_d \\
&= \mathbf{I}_1^T \tilde{\mathbf{E}}_{di} + \mathbf{p}_1^T \tilde{\mathbf{A}}_{di} + \mathbf{p}_2^T \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}_d - \lambda^T \mathbf{F}_d. \quad (20)
\end{aligned}$$

Through substituting the expressions of $\tilde{\mathbf{A}}_{ij}$, $\tilde{\mathbf{A}}_{\omega i}$, $\tilde{\mathbf{A}}_{di}$, $\tilde{\mathbf{E}}_{ij}$, $\tilde{\mathbf{E}}_{\omega i}$, $\tilde{\mathbf{E}}_{di}$, $\tilde{\mathbf{C}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{C}}_d$, $\tilde{\mathbf{H}}_i$, \mathbf{M}_j and \mathbf{N}_j into the above equalities, the following conditions hold:

$$\begin{aligned}
&\mathbf{I}_1^T \tilde{\mathbf{E}}_{ij} + \mathbf{p}_1^T \tilde{\mathbf{A}}_{ij} + \mathbf{p}_2^T \tilde{\mathbf{H}}_i \tilde{\mathbf{C}} + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times n}] \\
&= \mathbf{I}_1^T [\mathbf{C}_{fj} - \mathbf{E}_i \quad \mathbf{C}_{fj}] + [\mathbf{p}_{11}^T \quad \mathbf{p}_{12}^T] \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \mathbf{A}_{fj} - \mathbf{A}_i & \mathbf{A}_{fj} \end{bmatrix} \\
&\quad + \mathbf{p}_2^T [\mathbf{C}_i \quad \mathbf{C}_{\omega i} \quad \mathbf{C}_{di}] \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times n}] \\
&= [\mathbf{I}_1^T (\mathbf{C}_{fj} - \mathbf{E}_i) + \mathbf{p}_{11}^T \mathbf{A}_i + \mathbf{p}_{12}^T (\mathbf{A}_{fj} - \mathbf{A}_i) + \mathbf{p}_2^T \mathbf{C}_i + \lambda^T \mathbf{F}_d \\
&\quad \mathbf{p}_{12}^T \mathbf{A}_{fj} + \mathbf{I}_1^T \mathbf{C}_{fj}], \quad (21)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_1^T \mathbf{N}_j + \mathbf{p}_1^T \mathbf{M}_j - \mathbf{p}_2^T \\
&= \mathbf{I}_1^T \mathbf{D}_{fj} + [\mathbf{p}_{11}^T \quad \mathbf{p}_{12}^T] \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{fj} \end{bmatrix} - \mathbf{p}_2^T \\
&= \mathbf{I}_1^T \mathbf{D}_{fj} + \mathbf{p}_{12}^T \mathbf{B}_{fj} - \mathbf{p}_2^T, \quad (22)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_1^T \tilde{\mathbf{E}}_{\omega i} - \gamma \mathbf{I}_2^T + \mathbf{p}_1^T \tilde{\mathbf{A}}_{\omega i} + \mathbf{p}_2^T \tilde{\mathbf{H}}_i \tilde{\mathbf{D}} \\
&= -\mathbf{I}_1^T \tilde{\mathbf{E}}_{\omega i} - \gamma \mathbf{I}_2^T + [\mathbf{p}_{11}^T \quad \mathbf{p}_{12}^T] \begin{bmatrix} \mathbf{A}_{\omega i} \\ -\mathbf{A}_{\omega i} \end{bmatrix} \\
&\quad + \mathbf{p}_2^T [\mathbf{C}_i \quad \mathbf{C}_{\omega i} \quad \mathbf{C}_{di}] \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \\
&= -\mathbf{I}_1^T \tilde{\mathbf{E}}_{\omega i} - \gamma \mathbf{I}_2^T + (\mathbf{p}_{11}^T - \mathbf{p}_{12}^T) \mathbf{A}_{\omega i} + \mathbf{p}_2^T \mathbf{C}_{\omega i}, \quad (23)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_1^T \tilde{\mathbf{E}}_{di} + \mathbf{p}_1^T \tilde{\mathbf{A}}_{di} + \mathbf{p}_2^T \tilde{\mathbf{H}}_i \tilde{\mathbf{C}}_d - \lambda^T \mathbf{F}_d \\
&= -\mathbf{I}_1^T \tilde{\mathbf{E}}_{di} + [\mathbf{p}_{11}^T \quad \mathbf{p}_{12}^T] \begin{bmatrix} \mathbf{A}_{di} \\ -\mathbf{A}_{di} \end{bmatrix} \\
&\quad + \mathbf{p}_2^T [\mathbf{C}_i \quad \mathbf{C}_{\omega i} \quad \mathbf{C}_{di}] \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} - \lambda^T \mathbf{F}_d \\
&= -\mathbf{I}_1^T \tilde{\mathbf{E}}_{di} + (\mathbf{p}_{11}^T - \mathbf{p}_{12}^T) \mathbf{A}_{di} + \mathbf{p}_2^T \mathbf{C}_{di} - \lambda^T \mathbf{F}_d. \quad (24)
\end{aligned}$$

In light of the above analysis, we can obtain the stability conditions as follows:

$$\begin{aligned}
&\mathbf{I}_1^T (\mathbf{C}_{fj} - \mathbf{E}_i) + \mathbf{p}_{11}^T \mathbf{A}_i + \mathbf{p}_{12}^T (\mathbf{A}_{fj} - \mathbf{A}_i) \\
&\quad + \mathbf{p}_2^T \mathbf{C}_i + \lambda^T \mathbf{F}_d \prec 0, \quad (25)
\end{aligned}$$

$$\mathbf{p}_{12}^T \mathbf{A}_{fj} + \mathbf{I}_1^T \mathbf{C}_{fj} \prec 0, \quad (26)$$

$$\mathbf{I}_1^T \mathbf{D}_{fj} + \mathbf{p}_{12}^T \mathbf{B}_{fj} - \mathbf{p}_2^T \prec 0, \quad (27)$$

$$-\mathbf{I}_1^T \tilde{\mathbf{E}}_{\omega i} - \gamma \mathbf{I}_2^T + (\mathbf{p}_{11}^T - \mathbf{p}_{12}^T) \mathbf{A}_{\omega i} + \mathbf{p}_2^T \mathbf{C}_{\omega i} \prec 0, \quad (28)$$

$$-\mathbf{I}_1^T \tilde{\mathbf{E}}_{di} + (\mathbf{p}_{11}^T - \mathbf{p}_{12}^T) \mathbf{A}_{di} + \mathbf{p}_2^T \mathbf{C}_{di} - \lambda^T \mathbf{F}_d \prec 0. \quad (29)$$

On the basis of the above inequalities, it can be seen that $\mathbf{p}_{12}^T \mathbf{A}_{fj}$ and $\mathbf{p}_{12}^T \mathbf{B}_{fj}$ are non-convex terms. To transform them into convex ones, $\mathbf{P}_{Af,j}$ and $\mathbf{P}_{Bf,j}$ are considered as follows:

$$\mathbf{P}_{Af,j} = [\mathbf{p}_{12k}^T \mathbf{A}_{fj,k}], \mathbf{P}_{Bf,j} = [\mathbf{p}_{12k}^T \mathbf{B}_{fj,k}], \quad (30)$$

where $\mathbf{A}_{fj,k}$ and $\mathbf{B}_{fj,k}$ are the k -th row of \mathbf{A}_{fj} and \mathbf{B}_{fj} respectively. \mathbf{p}_{12k} is the k -th element in \mathbf{p}_{12} .

For example, we assume:

$$\begin{aligned}
\mathbf{A}_{fj} &= \begin{bmatrix} a_{fj,11} & a_{fj,12} & a_{fj,13} \\ a_{fj,21} & a_{fj,22} & a_{fj,23} \\ a_{fj,31} & a_{fj,32} & a_{fj,33} \end{bmatrix}, \\
\mathbf{p}_{12} &= [p_{121} \quad p_{122} \quad p_{123}]^T. \quad (31)
\end{aligned}$$

Then, we have:

$$\mathbf{P}_{Af,j} = \begin{bmatrix} p_{121} a_{fj,11} & p_{121} a_{fj,12} & p_{121} a_{fj,13} \\ p_{122} a_{fj,21} & p_{122} a_{fj,22} & p_{122} a_{fj,23} \\ p_{123} a_{fj,31} & p_{123} a_{fj,32} & p_{123} a_{fj,33} \end{bmatrix}. \quad (32)$$

Thereby, it can be obtained $\mathbf{p}_{12}^T \mathbf{A}_{fj} = \sum_{k=1}^n [\mathbf{P}_{Af,jk}]$, where $\mathbf{P}_{Af,jk}$ is the k -th row of $\mathbf{P}_{Af,j}$.

Taking (30) into (25)-(27), the following conditions hold:

$$\begin{aligned}
&\mathbf{I}_1^T (\mathbf{C}_{fj} - \mathbf{E}_i) + \mathbf{p}_{11}^T \mathbf{A}_i + \sum_{k=1}^n [\mathbf{P}_{Af,jk}] - \mathbf{p}_{12}^T \mathbf{A}_i \\
&\quad + \mathbf{p}_2^T \mathbf{C}_i + \lambda^T \mathbf{F}_d \prec 0, \quad (33)
\end{aligned}$$

$$\mathbf{I}_1^T \mathbf{C}_{fj} + \sum_{k=1}^n [\mathbf{P}_{Af,jk}] \prec 0, \quad (34)$$

$$\mathbf{I}_1^T \mathbf{D}_{fj} + \sum_{k=1}^n [\mathbf{P}_{Bf,jk}] - \mathbf{p}_2^T \prec 0. \quad (35)$$

Now, we are at the point that the convex stability conditions have been obtained. However, different from general systems, the positivity of the positive T-S fuzzy filter error system should be ensured as well. Thus, in the following, the positivity analysis will be of concern.

As mentioned in Remark 1 and Remark 2, the positivity conditions are shown as follows:

$$\mathbf{A}_{fj} \text{ is Metzler, } \mathbf{B}_{fj} \succeq 0, \mathbf{C}_{fj}(\mathbf{x}_f) \succeq 0, \mathbf{D}_{fj} \succeq 0, \forall j, \quad (36)$$

$$\begin{aligned} \hat{\mathbf{A}}_{ij} \text{ is Metzler, } \hat{\mathbf{A}}_{\omega i} \succeq 0, \hat{\mathbf{A}}_{di} \succeq 0, \\ \hat{\mathbf{L}}_{ij} \succeq 0, \hat{\mathbf{L}}_{\omega i} \succeq 0, \hat{\mathbf{L}}_{di} \succeq 0, \forall i, j \end{aligned} \quad (37)$$

Because the open-loop system is a positive system, therefore, the positive condition (37) can be ensured by satisfying the positive condition (36) and the following inequalities:

$$\mathbf{A}_{fj} + \mathbf{B}_{fj} \mathbf{C}_i - \mathbf{A}_i \succeq 0, \forall i, j, \quad (38)$$

$$\mathbf{B}_{fj} \mathbf{C}_{\omega i} - \mathbf{A}_{\omega i} \succeq 0, \forall i, j, \quad (39)$$

$$\mathbf{B}_{fj} \mathbf{C}_{di} - \mathbf{A}_{di} \succeq 0, \forall i, j, \quad (40)$$

$$\mathbf{D}_{fj} \mathbf{C}_i - \mathbf{E}_i + \mathbf{C}_{fj} \succeq 0, \forall i, j, \quad (41)$$

$$\mathbf{D}_{fj} \mathbf{C}_{\omega i} - \mathbf{E}_{\omega i} \succeq 0, \forall i, j, \quad (42)$$

$$\mathbf{D}_{fj} \mathbf{C}_{di} - \mathbf{E}_{di} \succeq 0, \forall i, j. \quad (43)$$

On account of $\mathbf{p}_{12} \succeq 0$, we multiply both sides of the (38) - (40) by \mathbf{p}_{12}^T , respectively, so as to get convex positivity conditions, which are presented as follows:

$$\mathbf{P}_{Af,j} + \mathbf{P}_{Bf,j} \mathbf{C}_i - [\mathbf{p}_{12k}^T \mathbf{A}_{i,k}] \succeq 0, \forall i, j, \quad (44)$$

$$\mathbf{P}_{Bf,j} \mathbf{C}_{\omega i} - [\mathbf{p}_{12k}^T \mathbf{A}_{\omega i,k}] \succeq 0, \forall i, j, \quad (45)$$

$$\mathbf{P}_{Bf,j} \mathbf{C}_{di} - [\mathbf{p}_{12k}^T \mathbf{A}_{di,k}] \succeq 0, \forall i, j, \quad (46)$$

where $\mathbf{A}_{i,k}$, $\mathbf{A}_{\omega i,k}$ and $\mathbf{A}_{di,k}$ are the k -th row of \mathbf{A}_i , $\mathbf{A}_{\omega i}$ and \mathbf{A}_{di} , respectively. \mathbf{p}_{12k} is the k -th element in \mathbf{p}_{12} .

For facilitating the analysis, we define:

$$\tilde{\mathbf{F}}_{1ij} = \mathbf{P}_{Af,j} + \mathbf{P}_{Bf,j} \mathbf{C}_i - [\mathbf{p}_{12k}^T \mathbf{A}_{i,k}], \quad (47)$$

$$\tilde{\mathbf{F}}_{2ij} = \mathbf{P}_{Bf,j} \mathbf{C}_{\omega i} - [\mathbf{p}_{12k}^T \mathbf{A}_{\omega i,k}], \quad (48)$$

$$\tilde{\mathbf{F}}_{3ij} = \mathbf{P}_{Bf,j} \mathbf{C}_{di} - [\mathbf{p}_{12k}^T \mathbf{A}_{di,k}], \quad (49)$$

$$\tilde{\mathbf{F}}_{4ij} = \mathbf{D}_{fj} \mathbf{C}_i - \mathbf{E}_i + \mathbf{C}_{fj}, \quad (50)$$

$$\tilde{\mathbf{F}}_{5ij} = \mathbf{D}_{fj} \mathbf{C}_{\omega i} - \mathbf{E}_{\omega i}, \quad (51)$$

$$\tilde{\mathbf{F}}_{6ij} = \mathbf{D}_{fj} \mathbf{C}_{di} - \mathbf{E}_{di}, \quad (52)$$

$$\begin{aligned} \tilde{\mathbf{Q}}_{1ij} &= \mathbf{I}_1^T (\mathbf{C}_{fj} - \mathbf{E}_i) + \mathbf{p}_{11}^T \mathbf{A}_i + \sum_{k=1}^n [\mathbf{P}_{Af,jk}] - \mathbf{p}_{12}^T \mathbf{A}_i \\ &+ \mathbf{p}_2^T \mathbf{C}_i + \lambda^T \mathbf{F}_d, \end{aligned} \quad (53)$$

$$\tilde{\mathbf{Q}}_{2j} = \sum_{k=1}^n [\mathbf{P}_{Af,jk}] + \mathbf{I}_1^T \mathbf{C}_{fj}, \quad (54)$$

$$\tilde{\mathbf{Q}}_{3j} = \mathbf{I}_1^T \mathbf{D}_{fj} + \sum_{k=1}^n [\mathbf{P}_{Bf,jk}] - \mathbf{p}_2^T, \quad (55)$$

$$\tilde{\mathbf{Q}}_{4i} = -\mathbf{I}_1^T \mathbf{E}_{\omega i} - \gamma \mathbf{I}_2^T + (\mathbf{p}_{11}^T - \mathbf{p}_{12}^T) \mathbf{A}_{\omega i} + \mathbf{p}_2^T \mathbf{C}_{\omega i}, \quad (56)$$

$$\tilde{\mathbf{Q}}_{5i} = -\mathbf{I}_1^T \mathbf{E}_{di} + (\mathbf{p}_{11}^T - \mathbf{p}_{12}^T) \mathbf{A}_{di} + \mathbf{p}_2^T \mathbf{C}_{di} - \lambda^T \mathbf{F}_d. \quad (57)$$

Remark 3: It is worth noting that the obtained results are very conservative because of the absence of the information of MFs. In general, researchers tend to use some mature techniques, like PDC technique and free-weighting matrix approach, to cut down the conservativeness of results instead of utilizing the information of MFs because the MFs will make stability analysis complex and MFD analysis is still in its early research stage. Thereby, taking the information of MFs into account for filter design of positive T-S fuzzy systems with time delay is very challenging but meaningful. In the following work, we will try our best to investigate the relaxed stability and positivity analysis for positive T-S fuzzy filter error systems.

C. Relaxed Positivity and Stability Analysis by using Piecewise-Linear Membership Functions

In this section, the PLMFs approximating method which has been explained in detail in [53], [59] is considered to extract the shape and boundary information of the original MFs. Meanwhile, the useful information of MFs is carried by some slack matrices to the positivity and stability analysis for relaxing the results.

According to (14), if the PLMFs method is considered, we obtain that $J < 0$ can be ensured by the following conditions:

$$\begin{aligned} \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{I}_1^T \tilde{\mathbf{L}}_{ij} + \mathbf{p}^T \tilde{\mathbf{A}}_{ij} + \lambda^T \mathbf{F}_d [\mathbf{I}_{n \times n} \quad \mathbf{0}_{n \times (n+l)}] \right) \\ \prec 0, \forall i, j; \end{aligned} \quad (58)$$

$$\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{I}_1^T \tilde{\mathbf{L}}_{\omega i} + \mathbf{p}^T \tilde{\mathbf{A}}_{\omega i} - \gamma \mathbf{I}_2^T \right) \prec 0, \forall i, j; \quad (59)$$

$$\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{I}_1^T \tilde{\mathbf{L}}_{di} + \mathbf{p}^T \tilde{\mathbf{A}}_{di} - \lambda^T \mathbf{F}_d \right) \prec 0, \forall i, j. \quad (60)$$

Next, combining with the analysis from (18) to (56), we can obtain the convex stability conditions with MFs as follows:

$$\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{Q}_{1ij} \right) \prec 0, \forall i, j; \quad (61)$$

$$\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{Q}_{2j} \right) \prec 0, \forall j; \quad (62)$$

$$\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{Q}_{3j} \right) \prec 0, \forall j; \quad (63)$$

$$\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{Q}_{4i} \right) \prec 0, \forall i; \quad (64)$$

$$\sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \left(\mathbf{Q}_{5i} \right) \prec 0, \forall i. \quad (65)$$

In order to cope with the cross term $\mu_i \eta_j$ in above stability conditions, the PLMFs method are introduced in the following. Because μ_i is related to \mathbf{x} but η_j is related to \mathbf{x}_f , hence, we

define $\hat{\mathbf{x}} = [\mathbf{x}; \mathbf{x}_f] \in \mathbb{R}^{2n}$, and $\hat{x}_1 = x_1, \hat{x}_2 = x_2, \dots, \hat{x}_n = x_n, \hat{x}_{n+1} = x_{f1}, \hat{x}_{n+2} = x_{f2}, \dots, \hat{x}_{2n} = x_{fn}$. Then dividing the whole operate domain into S sub-domains, where the sub-domain s is characterized by $\hat{x}_{l\text{sdmin}} \leq \hat{x}_{ls} \leq \hat{x}_{l\text{smax}}, l \in \{1, 2, \dots, 2n\}$ and $s \in \{1, 2, \dots, S\}$.

On the basis of the above definition, the cross term $\mu_i \eta_j$ can be replaced by the following expression:

$$\mu_i \eta_j = h_{ijs}(\hat{\mathbf{x}}) + \Delta_{ijs}(\hat{\mathbf{x}}), \forall i, j, s, \quad (65)$$

where $h_{ijs}(\hat{\mathbf{x}})$ is the approximated function and $\Delta_{ijs}(\hat{\mathbf{x}})$ is the approximation error.

The approximated function $h_{ijs}(\hat{\mathbf{x}})$ is expressed as follows:

$$h_{ijs}(\hat{\mathbf{x}}) = \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{2n}=1}^2 V_{1i_1s}(\hat{x}_1) V_{2i_2s}(\hat{x}_2) \cdots V_{2ni_{2n}s}(\hat{x}_{2n}) \times \mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) \forall i, j, s, \quad (66)$$

where $D_{i_1 i_2 \dots i_{2n}s} = [\hat{x}_{1i_1} \hat{x}_{2i_2} \cdots \hat{x}_{2ni_{2n}}]$. $\hat{x}_{li} = \hat{x}_{l\text{dmin}}$ if $i_l = 1$, and $\hat{x}_{li} = \hat{x}_{l\text{dmax}}$ if $i_l = 2, \forall l, s$. $V_{li_s}(\hat{x}_l)$ satisfies $\sum_{i_l=1}^2 V_{li_s}(\hat{x}_l) = 1, \forall l, s$.

Introducing (65) and (66) into the stability conditions (60) to (64), the relaxed stability conditions will be obtained. Because the derivations of relaxed stability conditions for (60) to (64) are similar, hence, **we only show the derivation of relaxed results for** (60). (61) to (64) will follow the same analysis.

$$\begin{aligned} \sum_{i=1}^p \sum_{j=1}^c \mu_i \eta_j \mathbf{Q}_{1ij} &= \sum_{i=1}^p \sum_{j=1}^c (h_{ijs}(\hat{\mathbf{x}}) + \Delta_{ijs}(\hat{\mathbf{x}})) \mathbf{Q}_{1ij} \\ &= \sum_{i=1}^p \sum_{j=1}^c (h_{ijs}(\hat{\mathbf{x}}) + \Delta_{ijs}(\hat{\mathbf{x}}) - \gamma_{ijs} + \gamma_{ijs}) \mathbf{Q}_{1ij} \\ &\leq \sum_{i=1}^p \sum_{j=1}^c \left((h_{ijs}(\hat{\mathbf{x}}) + \gamma_{ijs}) \mathbf{Q}_{1ij} + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{1ij} \right) \\ &= \sum_{i=1}^p \sum_{j=1}^c \left(\left(\sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{2n}=1}^2 V_{1i_1s}(\hat{x}_1) V_{2i_2s}(\hat{x}_2) \cdots V_{2ni_{2n}s}(\hat{x}_{2n}) \mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) + \gamma_{ijs} \right) \mathbf{Q}_{1ij} \right. \\ &\quad \left. + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{1ij} \right) \\ &= \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{2n}=1}^2 V_{1i_1s}(\hat{x}_1) V_{2i_2s}(\hat{x}_2) \cdots V_{2ni_{2n}s}(\hat{x}_{2n}) \\ &\quad \left(\sum_{i=1}^p \sum_{j=1}^c \left(\left(\mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) + \gamma_{ijs} \right) \mathbf{Q}_{1ij} \right. \right. \\ &\quad \left. \left. + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{1ij} \right) \right), \forall i, j, d, \quad (67) \end{aligned}$$

where γ_{ijs} and β_{ijs} are the lower and upper bound of $\Delta_{ijs}(\hat{\mathbf{x}})$, respectively, with satisfying $\gamma_{ijs} \leq \Delta_{ijs}(\hat{\mathbf{x}}) \leq \beta_{ijs}$. \mathbf{Y}_{1ij} is the slack matrix with satisfying $\mathbf{Y}_{1ij} \succeq 0, \mathbf{Y}_{1ij} \succeq \mathbf{Q}_{1ij}$.

Similarly, by introducing the slack matrices $\mathbf{Y}_{2j}, \mathbf{Y}_{3j}, \mathbf{Y}_{4i}$ and \mathbf{Y}_{5i} with satisfying $\mathbf{Y}_{2j} \succeq 0, \mathbf{Y}_{2j} \succeq \mathbf{Q}_{2j}, \mathbf{Y}_{3j} \succeq 0, \mathbf{Y}_{3j} \succeq \mathbf{Q}_{3j}, \mathbf{Y}_{4i} \succeq 0, \mathbf{Y}_{4i} \succeq \mathbf{Q}_{4i}, \mathbf{Y}_{5i} \succeq 0, \mathbf{Y}_{5i} \succeq \mathbf{Q}_{5i}$, respectively, the PLMFs method can be used for relaxing $\mathbf{Q}_{2j} -$

\mathbf{Q}_{5i} . For the sake of simplicity, we will not show the derivation of these relaxed conditions. In terms of above analysis, the relaxed results are **summarized** in the following theorem.

Theorem 1: Given a positive T-S fuzzy model with time delay (2) and satisfying Lemma 1, a positive T-S fuzzy filter (4) exists such that the stability and positivity of the positive T-S fuzzy filter error system (5) can be ensured with satisfying performance index (6), if there exist slack matrices $\mathbf{Y}_{1ij} \succeq 0, \mathbf{Y}_{2j} \succeq 0, \mathbf{Y}_{3j} \succeq 0, \mathbf{Y}_{4i} \succeq 0, \mathbf{Y}_{5i} \succeq 0$ as well as vectors $\mathbf{p}_{11} \succeq 0, \mathbf{p}_{12} \succeq 0, \lambda \succeq 0, \mathbf{p}_2$ and filter gain matrices $\mathbf{P}_{Bf,j} \succeq 0, \mathbf{C}_{fj} \succeq 0, \mathbf{D}_{fj} \succeq 0$ and Metzler matrix $\mathbf{P}_{Af,j}$ satisfying:

$$F_{1ij,rk} \text{ is SOS}, \forall i, j, r, k; \quad (68)$$

$$F_{2ij,rk} \text{ is SOS}, \forall i, j, r, k; \quad (69)$$

$$F_{3ij,rk} \text{ is SOS}, \forall i, j, r, k; \quad (70)$$

$$F_{4ij,rk} \text{ is SOS}, \forall i, j, r, k; \quad (71)$$

$$F_{5ij,rk} \text{ is SOS}, \forall i, j, r, k; \quad (72)$$

$$F_{6ij,rk} \text{ is SOS}, \forall i, j, r, k; \quad (73)$$

$$\rho^T \left(\text{diag}(\mathbf{Y}_{1ij} - \mathbf{Q}_{1ij}) \right) \rho \text{ is SOS}, \forall i, j; \quad (74)$$

$$\rho^T \left(\text{diag}(\mathbf{Y}_{2j} - \mathbf{Q}_{2j}) \right) \rho \text{ is SOS}, \forall j; \quad (75)$$

$$\sigma^T \left(\text{diag}(\mathbf{Y}_{3j} - \mathbf{Q}_{3j}) \right) \sigma \text{ is SOS}, \forall j; \quad (76)$$

$$v^T \left(\text{diag}(\mathbf{Y}_{4i} - \mathbf{Q}_{4i}) \right) v \text{ is SOS}, \forall i; \quad (77)$$

$$\rho^T \left(\text{diag}(\mathbf{Y}_{5i} - \mathbf{Q}_{5i}) \right) \rho \text{ is SOS}, \forall i; \quad (78)$$

$$\begin{aligned} & - \rho^T \left(\text{diag} \left(\sum_{i=1}^p \sum_{j=1}^c \left(\left(\mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) + \gamma_{ijs} \right) \mathbf{Q}_{1ij} \right. \right. \right. \\ & \left. \left. \left. + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{1ij} + \epsilon_1 \mathbf{I}_n \right) \right) \right) \rho \text{ is SOS}, \forall i, j; \quad (79) \end{aligned}$$

$$\begin{aligned} & - \rho^T \left(\text{diag} \left(\sum_{i=1}^p \sum_{j=1}^c \left(\left(\mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) + \gamma_{ijs} \right) \mathbf{Q}_{2j} \right. \right. \right. \\ & \left. \left. \left. + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{2j} + \epsilon_2 \mathbf{I}_n \right) \right) \right) \rho \text{ is SOS}, \forall j; \quad (80) \end{aligned}$$

$$\begin{aligned} & - \sigma^T \left(\text{diag} \left(\sum_{i=1}^p \sum_{j=1}^c \left(\left(\mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) + \gamma_{ijs} \right) \mathbf{Q}_{3j} \right. \right. \right. \\ & \left. \left. \left. + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{3j} + \epsilon_3 \mathbf{I}_l \right) \right) \right) \sigma \text{ is SOS}, \forall j; \quad (81) \end{aligned}$$

$$\begin{aligned} & - v^T \left(\text{diag} \left(\sum_{i=1}^p \sum_{j=1}^c \left(\left(\mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) + \gamma_{ijs} \right) \mathbf{Q}_{4i} \right. \right. \right. \\ & \left. \left. \left. + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{4i} + \epsilon_4 \mathbf{I}_m \right) \right) \right) v \text{ is SOS}, \forall i; \quad (82) \end{aligned}$$

$$\begin{aligned} & - \rho^T \left(\text{diag} \left(\sum_{i=1}^p \sum_{j=1}^c \left(\left(\mu_i(D_{i_1 i_2 \dots i_{2n}s}) \eta_j(D_{i_1 i_2 \dots i_{2n}s}) + \gamma_{ijs} \right) \mathbf{Q}_{5i} \right. \right. \right. \\ & \left. \left. \left. + (\beta_{ijs} - \gamma_{ijs}) \mathbf{Y}_{5i} + \epsilon_5 \mathbf{I}_n \right) \right) \right) \rho \text{ is SOS}, \forall i; \quad (83) \end{aligned}$$

where $\mathbf{Y}_{1ij}, \mathbf{Y}_{2j}, \mathbf{Y}_{3j}, \mathbf{Y}_{4i}, \mathbf{Y}_{5i}; \mathbf{p}_{11}, \mathbf{p}_{12}, \mathbf{p}_2, \lambda; \mathbf{P}_{Bf,j},$

C_{fj} , D_{fj} and $P_{Af,j}$ are to be determined. $\rho \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^l$ and $v \in \mathbb{R}^m$ are arbitrary vectors, $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ and ϵ_5 are predefined positive scalars. $I_n \in \mathbb{R}^n$, $I_l \in \mathbb{R}^l$ and $I_m \in \mathbb{R}^m$ are vectors with all the elements being 1. $F_{1ij,rk}$, $F_{2ij,rk}$, $F_{3ij,rk}$, $F_{4ij,rk}$, $F_{5ij,rk}$ and $F_{6ij,rk}$ are the r -th row and k -th column element in F_{1ij} , F_{2ij} , F_{3ij} , F_{4ij} , F_{5ij} and F_{6ij} , respectively. Q_{1ij} , Q_{2j} , Q_{3j} , Q_{4i} and Q_{5i} can be found in (52)–(56). A_{fj} and B_{fj} can be calculated based on (30).

IV. SIMULATION EXAMPLE

In this section, an example is given to demonstrate the effectiveness of the designed positive fuzzy filter. As described in Assumption 3, in the following, a positive T-S fuzzy model is given in the form of (2) with three rules:

$$\begin{aligned} A_1 &= \begin{bmatrix} -2.84 & 1.42 \\ 0.26 & -1.3 \end{bmatrix}, A_2 = \begin{bmatrix} -1.01 & 1.55 \\ 0.38 & -1.95 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -1.51 & 0.36 \\ 0.16 & -1.48 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.01 & 0.00 \\ 0.01 & 0.08 \end{bmatrix}, \\ A_{d2} &= \begin{bmatrix} 0.01 & 0.00 \\ 0.02 & 0.06 \end{bmatrix}, A_{d3} = \begin{bmatrix} 0.01 & 0.00 \\ 0.03 & 0.07 \end{bmatrix}, \\ A_{\omega 1} &= \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix}, A_{\omega 2} = \begin{bmatrix} 0.01 \\ 0.09 \end{bmatrix}, A_{\omega 3} = \begin{bmatrix} 0.02 \\ 0.05 \end{bmatrix}, \\ x &= [x_1 \ x_2]^T, C_1 = [1.01 \ 1.01], \\ C_2 &= [4.45 \ 4.83], C_3 = [5.56 \ 5.82], \\ C_{\omega 1} &= 0.01, C_{\omega 2} = 0.02, C_{\omega 3} = 0.03, C_{d1} = [0.01 \ 0.01], \\ C_{d2} &= [0.01 \ 0.02], C_{d3} = [0.01 \ 0.03], \\ E_1 &= [5.93 \ 6.33], E_2 = [1.87 \ 3.41], \\ E_3 &= [2.69 \ 1.55], \\ E_{\omega 1} &= 0.01, E_{\omega 2} = 0.02, E_{\omega 3} = 0.02, E_{d1} = [0.01 \ 0.01], \\ E_{d2} &= [0.02 \ 0.01], E_{d3} = [0.02 \ 0.01], \end{aligned}$$

As previously mentioned, F_d is a given matrix whose elements satisfy $F_{d,rk} = \max\{a_{di,rk}\}$, where $a_{di,rk}$ is the r -th row and k -th column element of A_{di} and $F_{d,rk}$ is the r -th row and k -th column element of F_d . The disturbance is chosen as $\tilde{w}(t) = \beta e^{-t} |\cos(2t)|$, where β is a constant scalar which is given reasonably. The MFs of the positive T-S fuzzy model with time delay are chosen as $w_1(x_1) = 1 - \frac{1}{1+e^{-(x_1-9)/3}}$, $w_3(x_1) = \frac{1}{1+e^{-(x_1-11)/3}}$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$. And the MFs of positive T-S fuzzy filter are chosen as $m_1(x_{f1}) = 1 - \frac{1}{1+0.5e^{-(x_{f1}-10)}}$, $m_2(x_{f1}) = 1 - m_1(x_{f1})$. When the PLMFs method is considered, we pick out the sample points of $w_i(x_1)$ at $x_1 \in \{0, 4, 8, 12, 16, 20\}$, $i = 1, 2, 3$ and the sampled points of $m_j(x_{f1})$ at $x_{f1} \in \{0, 4, 8, 12, 16, 20\}$, $j = 1, 2$, which means the operating domain of positive T-S fuzzy model and the one of positive T-S fuzzy filter are divided into 5 sub-domains, respectively.

In terms of the Theorem 1, the optimal performance index is calculated as $\gamma = 3.4997$ and the filter gain matrices are given in the following. Furthermore, the time responses of x_1 , x_{f1} , x_2 , x_{f2} , z , z_f and \tilde{z} are acquired and shown in Figs. 1 to 8. In these figures, different time responses of estimation errors

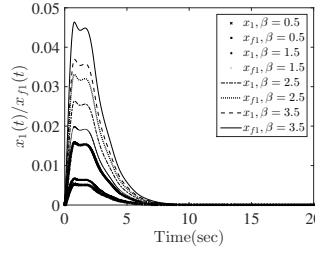


Fig. 1. System state $x_1(t)$ and filter state $x_{f1}(t)$ for different β

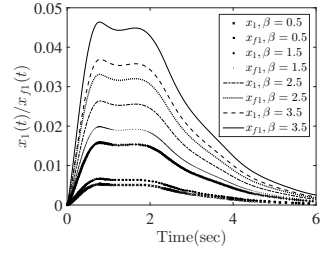


Fig. 2. Drawing of partial enlargement of system state $x_1(t)$ and filter state $x_{f1}(t)$ for different β

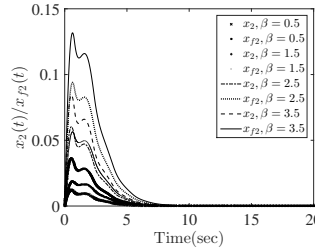


Fig. 3. System state $x_2(t)$ and filter state $x_{f2}(t)$ for different β

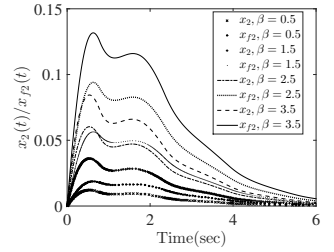


Fig. 4. Drawing of partial enlargement of system state $x_2(t)$ and filter state $x_{f2}(t)$ for different β

and state variables are displayed when the constant coefficient β in the disturbance are different.

$$\begin{aligned} A_{f1} &= \begin{bmatrix} -4.2012 & 3.6304 \times 10^{-2} \\ 2.6813 \times 10^{-4} & -11.4001 \end{bmatrix}, \\ A_{f2} &= \begin{bmatrix} -4.2095 & 3.2038 \times 10^{-2} \\ 2.2390 \times 10^{-4} & -11.4001 \end{bmatrix}, \\ B_{f1} &= \begin{bmatrix} 1.4010 \\ 10.0002 \end{bmatrix}, B_{f2} = \begin{bmatrix} 1.4002 \\ 10.0002 \end{bmatrix}, \\ C_{f1} &= [1.6538 \times 10^{-5} \ 4.0000 \times 10^{-1}], \\ C_{f2} &= [1.2540 \times 10^{-5} \ 3.9999 \times 10^{-1}], \\ D_{f1} &= 5.8713, D_{f2} = 5.8713. \end{aligned}$$

From the time response of the systems states and filter states, we can see that the positive T-S fuzzy filter can make the time response of the positive T-S fuzzy filter system with time delay quickly close to zero, which means the positive T-S fuzzy filter can achieve asymptotic stability and positivity of the positive T-S fuzzy filter error system. In addition, from Figs. 1 to 8, it can be seen that with the value of β becoming smaller, the time responses of x_{f1} , x_{f2} , z_f are closer to x_1 , x_2 , z , respectively. Hence, we can draw a conclusion that when the value of β is smaller, the filter effectiveness is better. Besides, when the disturbance parameter β gets bigger, the convergence rates of the filter state variables and system state variables slow down. And the estimation errors will be bigger with the disturbance becoming stronger. Therefore, it can be concluded that the smaller the disturbance parameter β , the better the convergence effect of the positive T-S fuzzy filter error system.

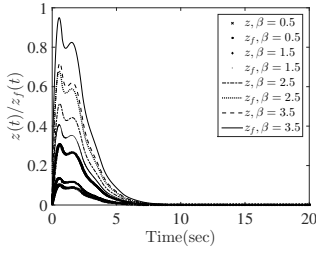


Fig. 5. Output $z(t)$ and the estimated output $z_f(t)$ for different β

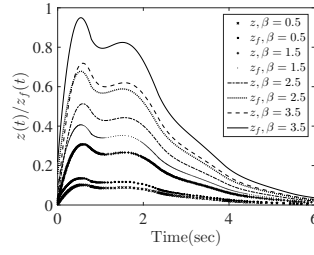


Fig. 6. Drawing of partial enlargement of output $z(t)$ and the estimated output $z_f(t)$ for different β

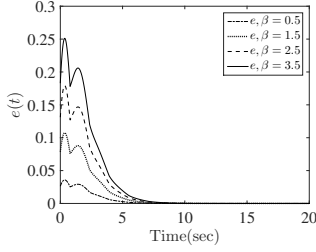


Fig. 7. Estimated errors $e(t)$ for different β

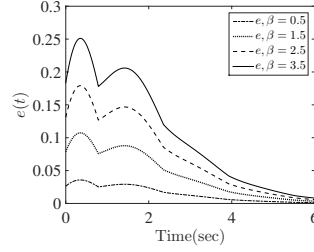


Fig. 8. Drawing of partial enlargement of estimated errors $e(t)$ for different β

V. CONCLUSION

In this paper, a positive T-S fuzzy filter has been designed for the positive T-S fuzzy system with time delay so that the stability and positivity analysis of the positive T-S fuzzy filter error system can be ensured under L_1 -induced performance. By employing the augmented approach and the linear copositive Lyapunov theory, the stability and positivity conditions with satisfying the optimal performance index γ have been obtained. Furthermore, considering the conservativeness, the advanced PLMFs method has been taken into account, which has a great help to explore some useful information inlaid in the MFs and facilitate the stability and positivity analysis. Ultimately, a simulation has been given to verify the validity of the strategy in this paper.

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